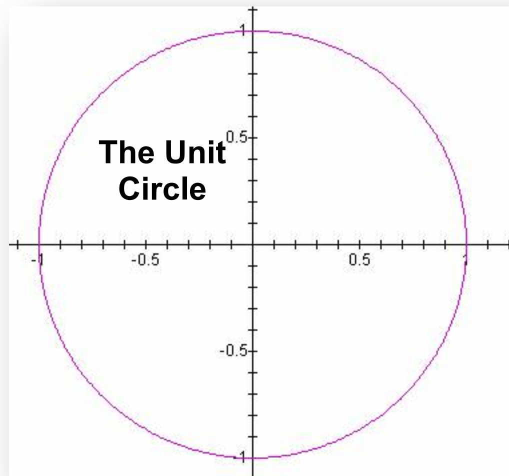
Algebra 2/Trigonometry Review
Sessions 1 & 2: Trigonometry Mega-Session

Trigonometry (Definition) - *The branch of mathematics that deals with the relationships between the sides and the angles of triangles and the calculations based on them, particularly the trigonometric functions.*

Trigonometry is based on two functions, **Sine** and **Cosine**. These functions are used in a variety of ways. The most basic application of these functions is finding missing dimensions of a right triangle when given limited information about the triangle. In Physics, sine and cosine are used to separate up/down motion (sine) and left/right motion (cosine).

Sine and Cosine can be easily derived from the unit circle. A **unit circle** has a center radius of one unit and its center is the origin of the xy plane.



Since the unit circle has a radius of 1 it crosses trough the exact points (1,0), (0,1), (-1,0) and (0,-1). These points are easy to notice in the picture to the right.

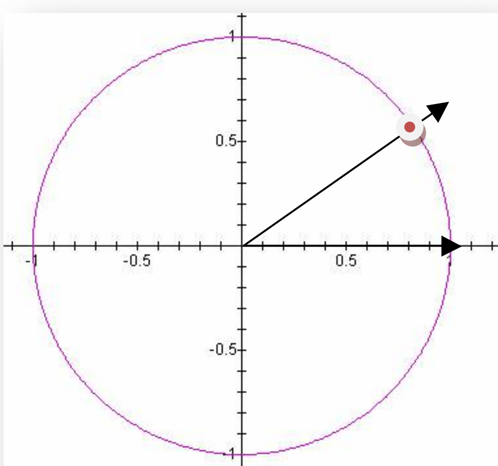
How many points are on the unit circle? Infinity.

It is an important concept to notice that every single x-value between -1 and 1 is located on the circle as well as every y-value between -1 and 1 also appears somewhere on the unit circle.

Some specific points are $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right),\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right),\left(\frac{-\sqrt{3}}{2},\frac{1}{2}\right)$.

We'll discuss how these points are determined and why they are important shortly.

Now let's take an acute angle of say 30° and put it in standard position. **Standard position** is where the vertex of the angle is on the origin, the initial side of the angle is the positive side of the x-axis and the terminal side of the angle would be somewhere on the unit circle (here it would be in the first quadrant as shown below).



When the angle is in standard position we notice that it crosses the unit circle at a specific point (the red dot). That point is the location of the sine and cosine of the angle.

The coordinates of the red dot would be:

$$(\cos(30°),\ \sin(30°))$$

So cosine is the x-value and sine is the y-value of the location the angle hits the unit circle when drawn in standard position.
We can now do some trig.

<u>Problem Set 1-1</u>: Find the value of each trigonometric function using your knowledge of the unit circle.

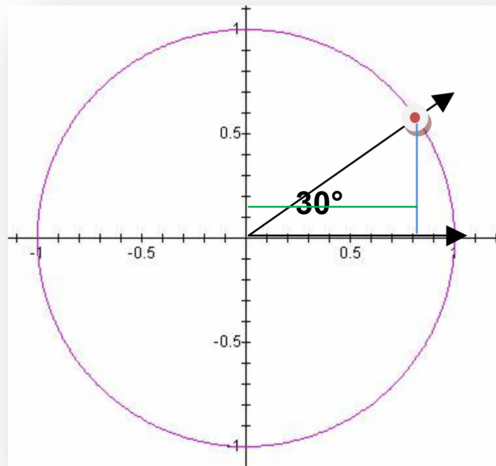
1. cos 0°   2. sin 0°   3. sin 90°   4. cos 180°

5. cos 270°   6. sin 270°   7. cos 90°   8. sin 180°

Special Right Triangles

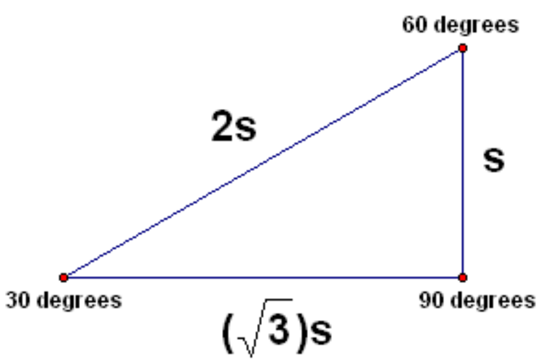
Now that we know the relationship between trigonometry and the unit circle we can get some triangles involved and use the Pythagorean Theorem to get some of those specific points on the unit circle that we mentioned earlier.

In the previous section we had a 30° angle in standard position and the red dot was $(\cos(30°), \sin(30°))$ but we want some exact values.



I've added a couple of line segments to our previous drawing, a blue vertical line segment and a green horizontal line segment.

Our knowledge from Part One tells us that the green value is the cosine of 30° and the blue measure is the sine of 30°. The nature of this triangle makes the lengths of the segments easy to find.



The triangle to the left uses Pythagorean Theorem to define the relationship between the sides of a **30-60-90** triangle. In our case the hypotenuse would be 1 (unit circle) so for the accompanying triangle set $= \frac{1}{2}$. This makes the blue segment $\frac{1}{2}$ and the green segment $\frac{\sqrt{3}}{2}$.

You will memorize these values so don't worry too much about memorizing the layouts of these triangles. Just remember that the short side of a triangle is ½ its hypotenuse.

You can "twist" this triangle so that the 60° angle is in standard position and you get two more values, cos 60° and sin 60°.

We should do a quick summary…

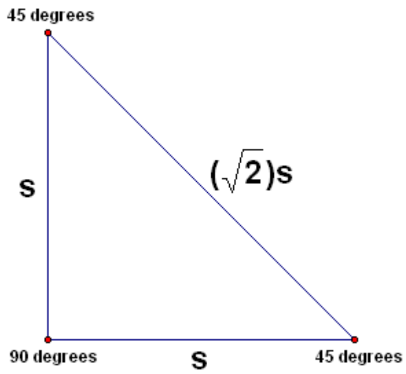
Problem Set 2-1: Find the exact values of each

1. sin 30°   2. cos 30°   3. sin 60°   4. cos 60°

Theorem: Complements of co-functions are equal.

Notice that sin 60 = cos 30 and vice versa.
We will define the term co-functions later but sine and cosine are one example.

There is only one other special right triangle that we need to acknowledge. **45-45-90**.



This will allow us to find the sine and cosine of 45°. And since our hypotenuse is again 1 we need to make $s = \frac{1}{\sqrt{2}}$, which is often written as $\frac{\sqrt{2}}{2}$. This is a little more confusing than the 30-60-90 triangle but it will just be memorized regardless.

So the sine and cosine of 45° is $\frac{\sqrt{2}}{2}$

Problem Set 2-2: Cumulative checkup. Find the exact values of each.

1. cos 270° 2. sin 45° 3. cos 60° 4. sin 90°

The Other Trig Functions

At this point we can put together our **Basic Trig Chart**. This should be memorized because it will be the basis for the first half of trigonometry.

	0°	30°	45°	60°	90°
Sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

You should be able to create this chart from memory. It only contains 5 different numbers and they are in ascending order for sine and descending order for cosine.

There are 4 other trigonometric functions.

Tangent and Cotangent

Tangent and cotangent are co-functions that are defined using quotients comprised of sine and cosine. Remember these "identities":

Tangent=Sine/Cosine and Cotangent=Cosine/Sine or tan θ=$\frac{\sin\ \theta}{\cos\ \theta}$ and cot θ=$\frac{\cos\ \theta}{\sin\ \theta}$

Note: The symbol θ is read as "theta". It is a Greek letter variable that is reserved for angle measure.

So you know the Tangent and Cotangent of a bunch of angles. Ex. tan 60°=$\frac{\sin\ 60°}{\cos\ 60°} = \frac{\frac{\sqrt{2}}{2}}{\frac{1}{2}} = \sqrt{3}$

And because tangent and cotangent are co-functions we know that cot 30°=$\sqrt{3}$.

What about 90°? Tan 90° =$\frac{1}{0}$ therefore Tangent is considered UNDEFINED at 90°.

Secant and Cosecant

Tangent and cotangent are co-functions that are defined by the reciprocals of sine and cosine. Remember these "identities" too:

Secant=1/Cosine and Cosecant=1/Sine or sec θ=$\frac{1}{\cos\ \theta}$ and csc θ=$\frac{1}{\sin\ \theta}$

Notice that we secant would be undefined at 90° and cosecant would be undefined at 0°. The rest of the values can be determined by flipping the associated trig function.

Ex. sec30°=$\frac{1}{\cos 30°} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$ So we just "flipped" cos 30° to get sec 30°

Notice that cosine goes with sine and secant goes with cosine. The C's go with the S's!

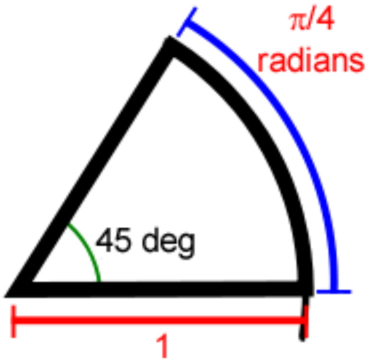
Sometimes the number $\frac{2}{\sqrt{3}}$ is rewritten to make it seem more ordinary. Some textbooks consider a radical in the denominator taboo. You can always fix it by multiplying the numerator and denominator by the radical. $\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{9}} = \frac{2\sqrt{3}}{3}$ This is an optional step.

<u>Problem Set 3-1:</u> Complete the trig chart below

	0°	30°	45°	60°	90°
Sin					
Cos					
Tan					
Cot					
Sec					
Csc					

Different Forms of Angle Measure

Up until this point angles were always presented to you in degree measure. It is very easy to visualize a 30° angle or even a 270° angle. But degree measure is inconvenient because it only works for angles. It isn't a true measure of length. Radian measure uses the distance between two specific sides of an angle to measure it.



If you take a point 1 unit on each ray of an angle, then the distance between the two points is the radian measure of that angle.

Therefore the radian measure of a 360° angle would be 2π because the distance (circumference) around a circle with radius 1 is 2π.

This makes a half circle π radians or 180°. This is the relationship we will use for converting from radians to degrees or vice-versa.

$$\pi \text{ rad} = 180°$$

This makes $1° = \frac{\pi}{180}$rad and $1 \text{ rad} = \frac{180°}{\pi}$

Therefore to change from….

- degrees to radians you multiply by $\frac{\pi}{180}$
- radians to degrees you multiply by $\frac{180}{\pi}$

It's easy to remember: To change to degrees leave 180 on top. To change to radians leave π on top.

Problem Set 3-2:

Convert to degrees: 1. $\frac{\pi}{2}$ 2. $\frac{5\pi}{6}$ 3. $\frac{4\pi}{3}$

Convert to Radians: 4. 30° 5. 45° 6. 300°

Problem Set 3-3: Find the exact values

1. $\sin\frac{\pi}{4}$ 2. $\sec\frac{\pi}{2}$ 3. $\tan\frac{\pi}{3}$

Minutes

On the June 2011 exam you were presented with this question:

The value of tan 126°43' to the *nearest ten-thousandth* is

(1) -1.3407 (3) -1.3548

(2) -1.3408 (4) -1.3549

The "43" represents a portion of the angle measured in Minutes. So we would read the angle as "126 degrees and 43 minutes". To evaluate this on your calculator you can use the minutes key in the ANGLE menu or you can convert 43 minutes to a fraction by dividing it by 60.

Extending the Trig Chart using Reference Angles

Not every angle is going to be acute or an exact multiple of 90. Often we are asked to find cos 225°, sin 330° or sec 135°. This is accomplished using the concept of reference angles and an important acronym: ASTC.

ASTC – A̲ll S̲tudents T̲ake C̲alculus - This tells you where the trig functions are positive. Write ASTC in quadrants I, II, III, and IV. **A**ll the trig functions are positive in QI. **S**ine is positive in QII (so is csc) the rest are negative. **T**an and cot are positive in QIII all others are negative. **C**os is positive in QIV (and sec) all others are negative.

Draw in your angle and drop a line to the x-axis. Find your reference angle and use your chart and ASTC to find the value.

Problem Set 4-1: Find the exact values

1. cos 225°   2. sin 330°   3. sec 135°

Problem Set 4-1 Continued….

1. sin $\frac{3\pi}{4}$ 2. sec $\frac{5\pi}{6}$ 3. tan $\frac{5\pi}{3}$

It would be foolish to try and memorize the entire trig chart. Just find values on an "as needed" basis.

Graphs of Trig Functions

The graphs of each trig functions have characteristics that you should be familiar with. Each trig function has a frequency and period and sine and cosine each have an amplitude. Let's look at each.

Function	Graph	Amplitude	Frequency	Period (Radians)
y=sin x				
y=cos x				
y=tan x				
y=cotx				
y= sec x				
y= csc x				

If you were to graph y=3sin2x you would see that the sine graph has been transformed. The 3 increases the amplitude to 3 and the 2 changes the frequency to 2, which reduces the period to π. *Period* = $\frac{2\pi}{Frequency}$

For y=a sin bx or y= a cos bx the amplitude is always |a|, the frequency is b and the period is then $\frac{2\pi}{b}$

The other trig functions can also be transformed but since they don't have amplitudes we don't usually list general rules. y=5tan2x would have a frequency of 4 and a period of $\frac{\pi}{2}$

Phase Shifts

If you recall function transformations from earlier mathematics, you'll remember that left/right and up/down movement is performed using addition and subtraction with x and y.

y=cos (x-90) + 2 will shift cosine right 90 degrees and up 2 units.

Practice 4-2: Transform sine to get cosine and then transform cosine to get sine.

Inverse Trig "Functions"

If you take the sin 30° you get the "height" of the sin curve when x=30 which is ½. If you were asked "the sine of what angle is ½?" would everyone say 30°? What about 150°? Doesn't that also have a sine of ½? Check on your calculator.

The function $\sin^{-1} x$ can correctly be called a function because it only gives you one of the angles that has a "height" of ½. It doesn't give the complete picture but it does provide a unique answer.

Problem Set 5-1: Evaluate each.

1. $\sin^{-1}\frac{\sqrt{2}}{2}$ 2. $\tan^{-1} -1$ 3. $\cos^{-1}\frac{\sqrt{3}}{2}$

4. Explain why $\sin^{-1} 2$ gives a calculator error.

Trig Equations

Solving for an angle: If you are solving for an angle (ex. $\sin\theta$ = ½) you need to use inverse sin (sometimes written as $\sin^{-1}\theta$ or called 2nd sin because of the calculator. This will only give you the REFERENCE angle you have to draw the reference angle in each possible quadrant (ASTC) and give each possible angle.

Ex. Solve $\sin\theta$ = ½ for all $0 \le \theta \le 360$.

a. Since sin is positive we are in QI or QII.
b. 2nd Sin ½ gives us 30˚ which is our reference angle.
c. Draw a 30˚ angle in QI and QII (drawn from x axis).
d. So answers are 30˚ and 150˚.

Note: There are infinite answers to $\sin\theta$ = ½ so we will always be given extra parameters.

Problem Set 5-2: Evaluate each.

1. $2\cos x = -1 \;\; 0 \le x \le 360$

2. Solve the following equation algebraically for all values of θ in the interval $0 \le \theta \le 180°$

$$2\sin\theta - 1 = 0$$

3. What are the values of θ in interval $0 \le \theta \le 360°$ that satisfy the equation $\tan\theta - \sqrt{3} = 0$?
   1) 60º, 240º
   2) 72º, 252º
   3) 72º, 108º, 252º, 288º
   4) 60º, 120º, 240º, 300º

4. Find the value of x in the interval $0 \le x \le 180°$ which satisfies the equation $cos^2 x -$ 2cosx=0.

5. Find the value of x in the interval $90° \le x \le 180°$ which satisfies the equation

$$\cos x - 2\cos x \sin x = 0$$

Trig Identities

Problem Set 5-3: Evaluate

1. $\cos 70° \cos 40° - \sin 70° \sin 40°$ is equivalent to
   1) $\cos 30°$
   2) $\cos 70°$
   3) $\cos 110°$
   4) $\sin 70°$

2. Which expression is equivalent to $\sin 22° \cos 18° + \cos 22° \sin 18°$?
   1) $\sin 4°$
   2) $\cos 4°$
   3) $\sin 40°$
   4) $\cos 40°$

3. Find the exact value of $\sin 75°$.

4. What is the value of $\sin 210° \cos 30° - \cos 210° \sin 30°$?

5. Express $\sin 75° \cos 15° - \cos 75° \sin 15°$ as a single trigonometric function of a positive acute angle.

Important Note: If you know either sine or cosine and which quadrant you are in then you can determine the associated co-function using the Pythagorean Trig Identity: $sin^2\theta + cos^2\theta=1$

6. If $sin\theta = \frac{3}{5}$ and θ is in quadrant II, find the other 5 trig functions.

7. If A and B are positive acute angles, $\sin A = 5/13$, and $\cos B = 4/5$, what is the value of $\sin(A+B)$?

1) 56/65	3) 33/65
2) 63/65	4) 16/65